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RIESZ TRANSFORM AND g-FUNCTION ASSOCIATED WITH BESSEL OPERATORS AND THEIR APPROPRIATE BANACH SPACES

BY

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ABSTRACT

We study g-functions and Riesz transforms related to the Bessel operators

$$\Delta_\mu = -x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2}.$$

The method we use allows us to characterize the Banach spaces $\mathbb{B}$ for which these operators are bounded when acting on $\mathbb{B}$-valued functions.

1. Introduction

In the descriptive and deep paper [6], Muckenhoupt and E. Stein defined and studied, in the context of orthogonal expansions, parallel objects to classical Fourier Analysis, namely conjugate functions, maximal functions, g-functions and multipliers. The technique involved the definition of the "harmonic extension" including a careful analysis of its kernel. Then they built a conjugate

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harmonic function and proved the existence of a boundary value function, the conjugate function. They got L^p boundedness of the conjugate function for p in the range $1 < p < \infty$ and some substitutive inequality in the case $p = 1$. This method was followed later by different authors when defining classical operators for orthogonal expansions. Five years later, E. Stein published his celebrated monograph [8] where the same objects were again treated. They were handled under a point of view (under our perception) based in a general analysis of a "Laplacian". He studied the "heat" and "Poisson" semigroups associated with the Laplacian and, from them, he derived the rest of the operators by using some spectral formulas.

In [1] this "semigroup approach", suggested in [8], is followed when the (positive) Laplacian is the Bessel operator

$$\Delta_\mu = -\frac{d^2}{dx^2} + \frac{\mu^2 - 1/4}{x^2}, \quad \text{where } \mu > -\frac{1}{2}. \tag{1.1}$$

The operator Δ_μ is self-adjoint in $L_2((0,\infty), dx)$ It can be written in divergence form as

$$\Delta_\mu = -x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} = A_\mu^* A_\mu,$$

being $A_\mu = x^{\mu+1/2} D x^{-\mu-1/2}$, and $A_\mu^* = x^{-\mu-1/2} D x^{\mu+1/2}$ the adjoint operator of A_μ. In [1], the Riesz transform $R_\mu = A_\mu \Delta_\mu^{-1/2}$ is defined and studied. It is shown, by means of the heavy use of some estimates from [6], that R_μ is in fact a principal value Calderón–Zygmund operator

$$R_\mu(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} R_\mu(x,y) f(y) dy, \qquad \text{a.e. } x \in (0,\infty), \quad f \in L^p(0,\infty), \quad 1 \le p < \infty. \tag{1.2}$$

R_μ being a Calderón–Zygmund operator, strong $L^p(0,\infty)$ inequalities for $1 < p < \infty$ and the classical weak type $(1,1)$ inequality are deduced. The method used in [1] is good enough when dealing with L^p inequalities for most operators associated with Δ_μ (e.g. g-function, Riesz potentials, multipliers). But that method did not fully exploit the structure of the operators, since it took into account neither the underlying measure space $((0,\infty))$ nor the similarities between the kernels of Δ_μ and the classical Laplacian. In this paper we present a method which take care of both facts. The core of the method could be summarized as follows: assume that a classical operator associated with Δ_μ is given by a kernel bounded by $|x-y|^{-1}$. Consider the kernel of the parallel operator associated with the classical Laplacian $-\Delta = -d^2/dx^2$. In the region $0 < x/2 < y < 2x$, the difference between both kernels is controlled by a positive function $L(x,y)$

which is the kernel of a bounded operator. Then the boundedness of the version of the operator associated either with Δ_μ or with $-\Delta$ implies the same property for the other version. The concrete statement of this vague idea is explained by Theorem 3.5.

In this paper we apply this method to characterize those Banach spaces $\mathbb{B}$ for which either the Riesz transforms (see Theorem 2.1) or Littlewood–Paley g-functions (see Theorems 2.4 and 2.5) are bounded when acting on $\mathbb{B}$-valued functions. Although we use the method explained above to obtain some geometrical properties of Banach spaces, we also obtain new results in the scalar-valued case (see Theorem 2.6).

We need to explain the extension of these operators to functions taking values in a Banach space $\mathbb{B}$. It is well known (and easy to check) that any positive bounded operator T on $L^p(\Omega)$ for all $p \in [1, \infty]$ naturally and boundedly extends to $L^p_{\mathbb{B}}(\Omega)$ for every Banach space $\mathbb{B}$, where $L^p_{\mathbb{B}}(\Omega)$ denotes the usual Bochner–Lebesgue L^p-space of $\mathbb{B}$-valued functions defined on Ω. Indeed, this is clear for $p = 1$ (via projective tensor product); the case $p = \infty$ is done by duality, and the range $1 < p < \infty$ by interpolation. With a slight abuse of notation (which will not cause any ambiguity), we shall denote these extensions still by the same symbol T. Concerning the Littlewood–Paley g-functions, we extend their definitions to $\mathbb{B}$-valued functions f by $(\int_0^\infty \|t\partial_t P_t f(x)\|^2_{\mathbb{B}} \frac{dt}{t})^{1/2}$ where it is understood that P_t, being linear and positive, has been extended to functions taking values in $\mathbb{B}$ as before. Since the Riesz transforms are linear, they extend in a natural way to the space $\mathbb{B} \otimes L^p(0, \infty)$ as $R_\mu(\sum b_i \varphi_i) = \sum b_i R_\mu(\varphi_i)$, $1 \leq p < \infty$.

The organization of the paper is the following. In section 2 we collect the main statements of the paper. Besides the results we described above, it is worth noting the characterization of the UMD property by using a kind of local Hilbert transform (see Theorem 2.2). Section 3 contains the statements and proofs of the technical theorems which we shall use in order to apply our method. The rest of the sections are devoted to the proofs (sometimes rather technical) of the main theorems in the paper.

Throughout this paper C always represents a suitable positive constant and it can change from one line to the other.

2. Preliminaries and statements of the results

In the euclidean case, it is known that the extension of the Hilbert transform is bounded from $L^p_{\mathbb{B}}(\mathbb{R}), 1 < p < \infty$ into itself or from $L^1_{\mathbb{B}}(\mathbb{R})$ into weak-$L^1_{\mathbb{B}}(\mathbb{R})$ if and only if $\mathbb{B}$ satisfies the so-called UMD property; see [3] and [4]. Moreover, a UMD space can be characterized by the almost everywhere convergence ($\varepsilon \to 0$) of the truncated integrals $\int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)dy$ for every function in $L^p_{\mathbb{B}}(\mathbb{R}), 1 \leq p < \infty$. We obtain the following parallel characterization of this property in terms of R_μ.

Theorem 2.1: *Let $\mathbb{B}$ be a Banach space and $\mu > -\frac{1}{2}$. Then the following conditions are equivalent:*

(i) *$\mathbb{B}$ is a UMD space.*

(ii) *For some (or equivalently, any) p, $1 < p < \infty$, R_μ can be extended as a bounded operator from $L^p_{\mathbb{B}}(0,\infty)$ into itself.*

(iii) *R_μ can be extended as a bounded operator from $L^1_{\mathbb{B}}(0,\infty)$ into $L^{1,\infty}_{\mathbb{B}}(0,\infty)$.*

(iv) *For any $f \in L^p_{\mathbb{B}}(0,\infty)$, $1 \leq p < \infty$, then for almost every $x \in (0,\infty)$, $R_\mu f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} R_\mu(x,y) f(y)dy \in \mathbb{B}$.*

On our way to prove this theorem, we shall get as a by-product of the method (in fact a corollary of Proposition 3.3) the following characterization of UMD spaces. We think it is essentially known, but it does not seem to be stated in the literature.

Theorem 2.2: *Let us define for functions $f \in \mathbb{B} \otimes L^p(\mathbb{R})$ the "local" Hilbert transform as*

$$H_{loc} f(x) = p.v. \int_{x/2}^{2x} \frac{f(y)}{x-y} dy, \quad x \in \mathbb{R},$$

and for $f \in \mathbb{B} \otimes L^p(0,\infty)$ the "partial" and "partial local" Hilbert transforms, respectively, as

$$\mathcal{H} f(x) = p.v. \int_0^\infty \frac{f(y)}{x-y} dy, \quad \mathcal{H}_{loc} f(x) = p.v. \int_{x/2}^{2x} \frac{f(y)}{x-y} dy, \quad x \in (0,\infty).$$

Then, the following statements are equivalent:

(a) *$\mathbb{B}$ has UMD property.*

(b) *$\mathcal{H}_{loc}$ is bounded from $L^p_{\mathbb{B}}(0,\infty)$ into itself for some (or, equivalently, for any) $p \in (1,\infty)$.*

(c) *$\mathcal{H}_{loc}$ is bounded from $L^1_{\mathbb{B}}(0,\infty)$ into $L^{1,\infty}_{\mathbb{B}}(0,\infty)$.*

(d) *For any $f \in L^1_{\mathbb{B}}(0,\infty)$, there exists $\mathcal{H}_{loc} f(x) \in \mathbb{B}$ for almost every $x \in (0,\infty)$*

The same equivalence holds with $\mathcal{H}$ in statements (b)–(d) in place of $\mathcal{H}_{loc}$, and also with H_{loc} and $\mathbb{R}$ in statements (b)–(d) instead of $\mathcal{H}_{loc}$ and $(0,\infty)$.

Let f be a function in $L^1(\mathbb{T})$, where $\mathbb{T}$ denotes the torus equipped with normalized Haar measure $d\theta$. The classical Littlewood–Paley g-function is defined for $z \in \mathbb{T}$ as

$$Gf(z) = \left(\int_0^1 (1-r)^2 \|\nabla P_r * f(z)\|^2 \frac{dr}{1-r} \right)^{1/2}.$$

In this notation,

$$\|\nabla P_r * f(t)\| = \left(|\frac{\partial P_r}{\partial r} * f(t)|^2 + |\frac{1}{r}\frac{\partial P_r}{\partial \theta} * f(t)|^2 \right)^{1/2},$$

where

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}$$

being the Poisson kernel for the disk. It is a classical result that for any $p \in (1,\infty)$ there exists a positive constant C_p such that

$$C_p^{-1}\|f\|_{L^p(\mathbb{T})} \le |\hat{f}(0)| + \|Gf\|_{L^p(\mathbb{T})} \le C_p\|f\|_{L^p(\mathbb{T})}.$$

When the functions f take values in a Banach space $\mathbb{B}$, this equivalence holds if and only if $\mathbb{B}$ is isomorphic to a Hilbert space. However, for a general Banach space, one of the two inequalities can be true. The study of these one-sided inequalities is the main objective of [10]. More generally, Stein introduced the following generalized "Littlewood–Paley g-function"

$$G^q f(z) = \left(\int_0^1 (1-r)^q \|\nabla P_r * f(z)\|_{\mathbb{B}}^q \frac{dr}{1-r} \right)^{1/q}.$$

Then $\mathbb{B}$ is said to be of Lusin cotype q (resp. Lusin type q) if there exist $p \in (1,\infty)$ and a positive constant C with $\|G^q f\|_{L^p(\mathbb{T})} \le C\|f\|_{L^p_{\mathbb{B}}(\mathbb{T})}$ (resp. $\|f\|_{L^p_{\mathbb{B}}(\mathbb{T})} \le C(\|\hat{f}(0)\|_{\mathbb{B}} + \|G^q f\|_{L^p(\mathbb{T})})$). It is not difficult to see that if $\mathbb{B}$ is of Lusin cotype q (resp. Lusin type q), then $2 \le q < \infty$ (resp. $1 < q \le 2$). It is proved in [10] that the definition above is independent of p; in other words, if one of the inequalities above holds for one $p \in (1,\infty)$, then so does it for every $p \in (1,\infty)$ (with a different constant depending on p). The main result of [10] states that a Banach space $\mathbb{B}$ is of Lusin type q (resp. Lusin cotype q) iff $\mathbb{B}$ is of martingale type q (resp. martingale cotype q). See [10] for the definition and references about the martingale type and cotype properties. In $\mathbb{R}$, the generalized "Littlewood–Paley g-function" can be defined for any $q \ge 1$ as

$$g^q(f)(x) = \left(\int_0^\infty t^q \|\nabla P_t * f(x)\|_{\ell^2_{\mathbb{B}}}^q \frac{dt}{t} \right)^{1/q},$$

where $\|\nabla P_t * f(x)\|_{\ell^2_{\mathbb{B}}} = (\|\partial_t P_t * f(x)\|_{\mathbb{B}}^2 + \|\partial_x P_t * f(x)\|_{\mathbb{B}}^2)^{1/2}$ and, here and in the sequel,

$$P_t(x) = \frac{1}{\pi}\frac{t}{|x|^2+t^2}$$

denotes the kernel of the Poisson semigroup for the upper half space. Then $\mathbb{B}$ is of martingale cotype q (resp. martingale type q) iff for some (equivalently every) $p \in (1, \infty)$ there is a constant C with

$$\|g^q(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \quad (\text{resp. } \|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leq C\|g^q(f)\|_{L^p(\mathbb{R}^n)}).$$

See [5]. Even more, the equivalence of the strong type (p, p) and weak type $(1, 1)$ are proved. Also, it is seen that it is equivalent to consider either g^q or the "partial" g-functions

$$g_t^q f(x) = \left(\int_0^\infty \|t\partial_t P_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}$$

and

$$g_x^q f(x) = \left(\int_0^\infty \|t\partial_x P_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{1/q}$$

to characterize Lusin type or cotype properties.

The results in [10] were extended to general markovian symmetric diffusion semigroups (see [5]) and a general Littlewood–Paley theory was developed for these semigroups. A markovian symmetric diffusion semigroup is a collection of linear operators $\{\mathcal{T}_t\}_{t\geq 0}$ defined on $\bigcup_p L^p(\Omega, d\mu)$ over a measure space $(\Omega, d\mu)$ satisfying the following properties: $\mathcal{T}_0 = \mathrm{Id}$, $\mathcal{T}_t\mathcal{T}_s = \mathcal{T}_{t+s}$, $\|\mathcal{T}_t\|_{L^p\to L^p} \leq 1$ for all $p \in [1, \infty]$, $\lim_{t\to 0} \mathcal{T}_t f = f$ in L^2, for all $f \in L^2$, $\mathcal{T}_t^* = \mathcal{T}_t$ on L^2, $\mathcal{T}_t f \geq 0$ if $f \geq 0$, $\mathcal{T}_t 1 = 1$. The corresponding Poisson $\mathcal{P}_t$ semigroup defined by subordination is again a symmetric diffusion semigroup; see [8].

Remark 2.3: Our Poisson semigroup does not fit in the theory developed in [5], since it is not markovian. The eigenfunctions of Δ_μ are $\{\varphi_y\}_{y>0}$, where, for every $y > 0$ (see [11, (6) and (7), p. 129]),

$$\varphi_y(x) = (yx)^{1/2} J_\mu(yx) \quad \text{and} \quad \Delta_\mu \varphi_y(x) = y^2 \varphi_y(x), \tag{2.1}$$

$J_\mu(z)$ being the Bessel function of the first kind of order μ. The Poisson kernel p_μ associated with Δ_μ is

$$p_\mu(t, x, y) = \int_0^\infty e^{-zt} \varphi_z(x)\varphi_z(y) dz, \quad t, x, y \in (0, \infty). \tag{2.2}$$

The corresponding Poisson integral was considered in [7] (and more recently in [2]). It is defined by

$$P_{\mu,t}(f)(x) = \int_0^\infty p_\mu(t, x, y) f(y) dy, \quad t, x \in (0, \infty),$$

where f is a suitable function. This Poisson semigroup is not Markovian. Indeed, since $\chi_{(0,\infty)} \in L^\infty(0,\infty)$, the function $u(t,x) = P_{\mu,t}(\chi_{(0,\infty)})(x)$ satisfies the Laplace type equation $\Delta_{\mu,x}u - \partial^2 u/\partial t^2 = 0$ ([2, Remark 2.5]). It is not hard to see that the function $v(t,x) = 1$, $t, x \in (0,\infty)$, is not a solution of the last partial differential equation.

Given a Banach space $\mathbb{B}$, in analogy with the classical case, for any $1 < q < \infty$ we define the generalized ("Bessel") g-function

$$g^q_\mu f(x) = \left(\int_0^\infty \|t\nabla P_{\mu,t} f(x)\|^q_{\ell^2_{\mathbb{B}}} \frac{dt}{t} \right)^{1/q},$$

where

$$\|\nabla P_{\mu,t}(f)(x)\|_{\ell^2_{\mathbb{B}}} = (\|\partial_t P_{\mu,t} f(x)\|^2_{\mathbb{B}} + \|A_{\mu,x} P_{\mu,t} f(x)\|^2_{\mathbb{B}})^{1/2}.$$

We will also consider the "partial" generalized g-functions

$$g^q_{\mu,t} f(x) = \left(\int_0^\infty \|t\partial_t P_{\mu,t} f(x)\|^q_{\mathbb{B}} \frac{dt}{t} \right)^{1/q}$$

and

$$g^q_{\mu,x} f(x) = \left(\int_0^\infty \|tA_{\mu,x} P_{\mu,t} f(x)\|^q_{\mathbb{B}} \frac{dt}{t} \right)^{1/q}.$$

We have the following parallel result to the euclidean case.

THEOREM 2.4: *Let $\mathbb{B}$ be a Banach space, $2 \le q < \infty$ and $\mu > -\frac{1}{2}$. Then the following statements are equivalent:*

(i) *$\mathbb{B}$ is of Lusin cotype q.*

(ii) *For every (or equivalently, for some) $p \in (1,\infty)$ there is a constant $C > 0$*

$$\|g^q_\mu(f)\|_{L^p((0,\infty),dx)} \le C\|f\|_{L^p_{\mathbb{B}}((0,\infty),dx)}, \quad \forall f \in L^p_{\mathbb{B}}((0,\infty),dx).$$

(iii) *There exists a constant $C > 0$ with*

$$\|g^q_\mu(f)\|_{L^{1,\infty}((0,\infty),dx)} \le C\|f\|_{L^1_{\mathbb{B}}((0,\infty),dx)}, \quad \forall f \in L^1_{\mathbb{B}}((0,\infty),dx).$$

(iv) *For any $f \in L^1_{\mathbb{B}}((0,\infty),dx)$, $g^q_\mu f(x) < \infty$ for almost every $x \in (0,\infty)$.*

The same equivalences hold with $g^q_{\mu,x}$ or $g^q_{\mu,t}$ instead of g^q_μ in (ii), (iii) and (iv).

The result concerning the Lusin type property of the space is as follows.

Theorem 2.5: *Let $\mathbb{B}$ be a Banach space, $1 < q \le 2$ and $\mu > -\frac{1}{2}$. The following statements are equivalent:*

(i) *$\mathbb{B}$ has Lusin type q.*

(ii) *For some (or equivalently, for any) $p \in (1,\infty)$, there exists $C > 0$ such that*

$$\|f\|_{L^p_{\mathbb{B}}(0,\infty)} \le C\|g^q_{\mu,t} f\|_{L^p(0,\infty)}.$$

As a consequence of Theorems 2.4 and 2.5 and the fact that $\mathbb{R}$ is of Lusin type 2 and Lusin cotype 2, we get the following result of independent interest.

Theorem 2.6: *Let $1 < p < \infty$ and $\mu > -\frac{1}{2}$. Then there exists a constant $C_p > 0$ such that*

$$C_p^{-1}\|f\|_{L^p(0,\infty)} \le \|g^2_\mu f\|_{L^p(0,\infty)} \le C_p\|f\|_{L^p(0,\infty)}.$$

3. Technical tools

The operators H_1 and H_2 will denote the classical Hardy operators

$$H_1 f(x) = \frac{1}{x}\int_0^x f(y)dy \quad \text{and its dual} \quad H_2 f(x) = \int_x^\infty \frac{1}{y} f(y)dy, \tag{3.1}$$

which are known to be bounded on $L^p(0,\infty)$, $1 < p < \infty$ and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$ (see [12, p. 20], for $1 < p < \infty$; the case $p = 1$ is clear.)

Definition 3.1: Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be a pair of Banach spaces, and $K(x,y)$ be a function defined in $\mathbb{R}\times\mathbb{R}$ with values in $\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)$. We say that an operator T is a principal value operator in $\mathbb{R}$ with associated kernel $K(x,y)$ if

$$Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x,y)f(y)dy, x\in\mathbb{R}, \text{ for } f \in \mathbb{B}_1 \otimes L^\infty_c(\mathbb{R}).$$

Definition 3.2: Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be a pair of Banach spaces, and $K(x,y)$ be a function defined in $(0,\infty)\times(0,\infty)$ with values in $\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)$. We say that an operator $\mathcal{T}_{\mathrm{loc}}$ is a local principal value operator in $(0,\infty)$ with associated kernel $K(x,y)$ if

$$\mathcal{T}_{\mathrm{loc}} f(x) = \lim_{\varepsilon\to 0}\mathcal{T}_{\varepsilon,\mathrm{loc}} f(x) = \lim_{\varepsilon\to 0}\int_{\substack{|x-y|>\varepsilon \\ 0<x/2<y<2x}} K(x,y)f(y)dy,$$
$$x\in(0,\infty), \quad \text{for } f\in\mathbb{B}_1\otimes L^\infty_c(0,\infty).$$

PROPOSITION 3.3: *Let $\mathbb{B}_1$, $\mathbb{B}_2$ be two Banach spaces and $K(x)$ be an odd or even $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-valued kernel, satisfying $\|K(x)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq C|x|^{-1}$. Assume the existence of the principal value operators T and $\mathcal{T}_{loc}$ with associated kernel $K(x-y)$ in $\mathbb{R}$ and in $(0,\infty)$, respectively. Then, the following hold:*

(a) *Let p be in the range $1 \leq p < \infty$. For any $f \in L^p_{\mathbb{B}_1}(\mathbb{R})$, there exists $Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x)$ a.e. $x \in \mathbb{R}$, if and only if for any $f \in L^p_{\mathbb{B}_1}(0,\infty)$, there exists $\mathcal{T}_{loc}f(x) = \lim_{\varepsilon\to 0} \mathcal{T}_{\varepsilon,loc}f(x)$ a.e. $x \in (0,\infty)$.*

(b) *Let p be in the range $1 < p < \infty$. T is bounded : $L^p_{\mathbb{B}_1}(\mathbb{R}) \longrightarrow L^p_{\mathbb{B}_2}(\mathbb{R})$ if and only if $\mathcal{T}_{loc}$ is bounded : $L^p_{\mathbb{B}_1}(0,\infty) \longrightarrow L^p_{\mathbb{B}_2}(0,\infty)$.*

(c) *T maps $L^1_{\mathbb{B}_1}(\mathbb{R})$ into $L^{1,\infty}_{\mathbb{B}_2}(\mathbb{R})$ boundedly if and only if $\mathcal{T}_{loc}$ maps $L^1_{\mathbb{B}_1}(0,\infty)$ into $L^{1,\infty}_{\mathbb{B}_2}(0,\infty)$ boundedly.*

Proof: Let $f \in L^\infty_{c,\mathbb{B}_1}(\mathbb{R})$ and denote $\tilde{f}(y) = f(-y)$. For every $\varepsilon > 0$ denote $K_\varepsilon(x) = K(x)\chi_{(\varepsilon,\infty)}(|x|)$. We can write

$$\begin{aligned}
T_\varepsilon f(x) &= \int_{-\infty}^{\infty} K_\varepsilon(x-y)f(y)dy \\
&= \int_{-\infty}^{0} K_\varepsilon(x-y)f(y)dy + \int_0^\infty K_\varepsilon(x-y)f(y)dy \\
&= \int_0^\infty K_\varepsilon(x+y)f(-y)dy + T_\varepsilon(f\chi_{(0,\infty)})(x) \\
&= \int_0^\infty \pm K_\varepsilon(-x-y)f(-y)dy + T_\varepsilon(f\chi_{(0,\infty)})(x) \qquad (3.2)\\
&= \pm\, T_\varepsilon(\tilde{f}\chi_{(0,\infty)})(-x) + T_\varepsilon(f\chi_{(0,\infty)})(x) \\
&= \pm\, T_\varepsilon(\tilde{f}\chi_{(0,\infty)})(-x)\chi_{(-\infty,0)}(x) \pm T_\varepsilon(\tilde{f}\chi_{(0,\infty)})(-x)\chi_{(0,\infty)}(x) \\
&\quad + T_\varepsilon(f\chi_{(0,\infty)})(x)\chi_{(-\infty,0)}(x) + T_\varepsilon(f\chi_{(0,\infty)})(x)\chi_{(0,\infty)}(x) \\
&= I + II + III + IV.
\end{aligned}$$

For the terms II and III observe that

$$\begin{aligned}
&\|T_\varepsilon(\tilde{f}\chi_{(0,\infty)})(-x)\chi_{(0,\infty)}(x)\|_{\mathbb{B}_2} \\
&\qquad \leq \int_0^\infty \|K_\varepsilon(-x-y)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|\tilde{f}(y)\|_{\mathbb{B}_1}\,dy\chi_{(0,\infty)}(x) \\
&\qquad = \int_0^\infty \|K_\varepsilon(x+y)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|\tilde{f}(y)\|_{\mathbb{B}_1}\,dy\chi_{(0,\infty)}(x) \\
&\qquad \leq \int_0^\infty \frac{C}{x+y}\|\tilde{f}(y)\|_{\mathbb{B}_1}\,dy\chi_{(0,\infty)}(x) \qquad (3.3)\\
&\qquad = \left(\int_0^x + \int_x^\infty\right)\frac{C}{x+y}\|\tilde{f}(y)\|_{\mathbb{B}_1}\,dy\chi_{(0,\infty)}(x)
\end{aligned}$$

$$\leq \left(\int_0^x \frac{C}{x} \|\tilde{f}(y)\|_{\mathbb{B}_1}\, dy + \int_x^\infty \frac{C}{y} \|\tilde{f}(y)\|_{\mathbb{B}_1}\, dy \right) \chi_{(0,\infty)}(x)$$
$$= C(H_1(\|\tilde{f}\chi_{(0,\infty)}\|_{\mathbb{B}_1})(x) + H_2(\|\tilde{f}\chi_{(0,\infty)}\|_{\mathbb{B}_1})(x))\chi_{(0,\infty)}(x),$$

where H_1 and H_2 are the Hardy operators defined in (3.1). By a similar calculation, we have

$$(3.4)\quad \|T_\varepsilon(f\chi_{(0,\infty)})(x)\chi_{(-\infty,0)}(x)\|_{\mathbb{B}_2} \leq \int_0^\infty \frac{C}{|x|+y} \|\tilde{f}(y)\|_{\mathbb{B}_1}\, dy \chi_{(-\infty,0)}(x)$$
$$\leq C(H_1(\|f\chi_{(0,\infty)}\|_{\mathbb{B}_1})(|x|) + H_2(\|f\chi_{(0,\infty)}\|_{\mathbb{B}_1})(|x|))\chi_{(-\infty,0)}(x).$$

In order to handle term IV, we write

$$T_\varepsilon(f\chi_{(0,\infty)})(x)\chi_{(0,\infty)}(x) = \int_{|x-y|>\varepsilon, 0<y<x/2} K(x-y)f(y)dy\chi_{(0,\infty)}(x)$$
$$+ \mathcal{T}_{\varepsilon,\mathrm{loc}}(f\chi_{(0,\infty)})(x)\chi_{(0,\infty)}(x)$$
$$+ \int_{|x-y|>\varepsilon, 2x<y} K(x-y)f(y)dy\chi_{(0,\infty)}(x).$$

By taking norms inside the integrals, the first and third terms in this sum verify

$$\left\| \int_{|x-y|>\varepsilon, 0<y<x/2} K(x-y)f(y)dy\chi_{(0,\infty)}(x) \right\|_{\mathbb{B}_2} + \left\| \int_{|x-y|>\varepsilon, 2x<y} K(x-y)f(y)dy\chi_{(0,\infty)}(x) \right\|_{\mathbb{B}_2}$$
$$\leq C(H_1(\|f\|_{\mathbb{B}_1})(x)\chi_{(0,\infty)}(x) + H_2(\|f\|_{\mathbb{B}_1})(x)\chi_{(0,\infty)}(x)).$$

We can proceed in the same way for I. Summarizing, we have obtained

$$T_\varepsilon f(x) = \pm \mathcal{T}_{\varepsilon,\mathrm{loc}}(\tilde{f}\chi_{(0,\infty)})(-x)\chi_{(-\infty,0)}(x)$$
$$+ \mathcal{T}_{\varepsilon,\mathrm{loc}}(f\chi_{(0,\infty)})(x)\chi_{(0,\infty)}(x) + Q_\varepsilon(f)(x),$$

where Q_ε is a sum of integral operators and

$$\|Q_\varepsilon(f)(x)\|_{\mathbb{B}_2} \leq C((H_1(\|f\chi_{(0,\infty)}\|_{\mathbb{B}_1})(x) + H_2(\|f\chi_{(0,\infty)}\|_{\mathbb{B}_1})(x))\chi_{(0,\infty)}(x)$$
$$+ (H_1(\|\tilde{f}\chi_{(0,\infty)}\|_{\mathbb{B}_1})(|x|) + H_2(\|\tilde{f}\chi_{(0,\infty)}\|_{\mathbb{B}_1})(|x|))\chi_{(-\infty,0)}(x))$$
$$= F(x).$$

Then, (a) follows. By taking limits in (3.5), we get

$$Tf(x) = \pm \mathcal{T}_{\mathrm{loc}}(\tilde{f}\chi_{(0,\infty)})(-x)\chi_{(-\infty,0)}(x)$$
$$+ \mathcal{T}_{\mathrm{loc}}(f\chi_{(0,\infty)})(x)\chi_{(0,\infty)}(x) + Q(f)(x),$$

where Q satisfies

$$\|Q(f)(x)\|_{\mathbb{B}_2} \leq F(x)$$

Then, (b) and (c) follow. ∎

As we mentioned in the previous section, the clearest example of an operator T in Proposition 3.3 is the case in which $\mathbb{B}_1 = \mathbb{B}_2$ and $K(x) = x^{-1}$. Then we get Theorem 2.2.

The g-functions g_t^q and g_x^q can be seen as vector-valued operators given by convolution kernels. In both cases, $\mathbb{B}_1 = \mathbb{B}$, $\mathbb{B}_2 = L^q_{\mathbb{B}}((0,\infty), dt/t)$. $g_t^q f(x) = \|t\partial_t P_t * f(t,x)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)}$ and $g_x^q f(x) = \|t\partial_x P_t * f(t,x)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)}$, where $t\partial_t P_t * f(x) = \int_{\mathbb{R}} K_1(t, x-y)f(y)dy$, and $t\partial_x P_t * f(x) = \int_{\mathbb{R}} K_2(t,x-y)f(y)dy$. Moreover, if $f \in \mathbb{B}_1 \otimes L_c^\infty(\mathbb{R})$, then

$$\int_{\mathbb{R}} K_i(t,x-y)f(y)dy = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K_i(t,x-y)f(y)dy, \quad i = 1,2.$$

It is easy to see that $K_1(t,-x) = K_1(t,x)$ and $K_2(t,-x) = -K_2(t,x)$. Moreover,

$$\begin{aligned} \|K_1(t,x)\|^q_{L^q_{\mathbb{B}}((0,\infty),dt/t)} &= \frac{1}{\pi}\int_0^\infty \left| t\frac{x^2-t^2}{(x^2+t^2)^2}\right|^q \frac{dt}{t} \\ &= \frac{1}{\pi|x|^q}\int_0^\infty \left| u\frac{1-u^2}{(1+u^2)^2}\right|^q \frac{du}{u} = \frac{C}{|x|^q}. \end{aligned} \tag{3.6}$$

The same holds for K_2 by an analogous calculation. Thus, the vector-valued operators given by K_i, $i = 1,2$ satisfy the hypothesis in Proposition 3.3.

The same arguments which lead to Proposition 3.3 easily give the following proposition.

Proposition 3.4: *Let $\mathbb{B}_1, \mathbb{B}_2$ be two Banach spaces and $K(x,y)$ be a $\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)$-valued kernel, satisfying $\|K(x,y)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq C|x-y|^{-1}$. Let us assume that there exist the principal value operators $\mathcal{T}$ and $\mathcal{T}_{loc}$ (see Definition 3.2) with associated kernel $K(x,y)$ in $(0,\infty)$ (both $\mathcal{T}$ and $\mathcal{T}_{loc}$). Then, the following hold:*

(a) *Let p in the range $1 \leq p < \infty$. For any $f \in L^p_{\mathbb{B}_1}(0,\infty)$, there exists $\mathcal{T}f(x) = \lim_{\varepsilon\to 0}\mathcal{T}_\varepsilon f(x)$ a.e. $x \in (0,\infty)$, if and only if for any $f \in L^p_{\mathbb{B}_1}(0,\infty)$ there exists $\mathcal{T}_{loc}f(x) = \lim_{\varepsilon\to 0}\mathcal{T}_{\varepsilon,loc}f(x)$ a.e. $x \in (0,\infty)$.*

(b) *Given $p \in (1,\infty)$, $\mathcal{T}$ maps $L^p_{\mathbb{B}_1}(0,\infty)$ into $L^p_{\mathbb{B}_2}(0,\infty)$ for $p \in (1,\infty)$ if and only if $\mathcal{T}_{loc}$ maps $L^p_{\mathbb{B}_1}(0,\infty)$ into $L^p_{\mathbb{B}_2}(0,\infty)$.*

(c) *$\mathcal{T}$ maps $L^1_{\mathbb{B}_1}(0,\infty)$ into $L^{1,\infty}_{\mathbb{B}_2}(0,\infty)$ boundedly if and only if $\mathcal{T}_{loc}$ maps $L^1_{\mathbb{B}_1}(0,\infty)$ into $L^{1,\infty}_{\mathbb{B}_2}(0,\infty)$ boundedly.*

Our method crystallizes in the following theorem.

THEOREM 3.5: *Let $\mathbb{B}_1$, $\mathbb{B}_2$ be a pair Banach spaces. Let us consider principal value operators T and S on $\mathbb{R}$ and $(0,\infty)$, respectively, with associated kernels $T(x-y)$ and $S(x,y)$, satisfying that $\|T(x-y)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq C|x-y|^{-1}$, $\|S(x,y)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \leq C|x-y|^{-1}$ and T being an odd or even function. Assume*

$$\|T(x-y)-S(x,y)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\chi_{(x/2,2x)}(y) \leq N(x,y), \quad \text{for } x>0,$$

where N is the kernel of a bounded operator on $L^p(0,\infty)$ for every $p \in [1,\infty)$. Then:

(a) *Given $p \in [1,\infty)$, for any $f \in L^p_{\mathbb{B}_1}(\mathbb{R})$, there exists $Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x)$ a.e. $x \in (0,\infty)$, if and only if for any $f \in L^p_{\mathbb{B}_1}(0,\infty)$, there exists $Sf(x) = \lim_{\varepsilon\to 0} S_\varepsilon f(x)$ a.e. $x \in (0,\infty)$.*

(b) *Let $1<p<\infty$. T maps $L^p_{\mathbb{B}_1}(\mathbb{R})$ into $L^p_{\mathbb{B}_2}(\mathbb{R})$ if and only if S maps $L^p_{\mathbb{B}_1}(0,\infty)$ into $L^p_{\mathbb{B}_2}(0,\infty)$.*

(c) *T maps $L^1_{\mathbb{B}_1}(\mathbb{R})$ into $L^{1,\infty}_{\mathbb{B}_2}(\mathbb{R})$ boundedly if and only if S maps $L^1_{\mathbb{B}_1}(0,\infty)$ into $L^{1,\infty}_{\mathbb{B}_2}(0,\infty)$ boundedly.*

Proof: The theorem is a consequence of Propositions 3.3 and 3.4. ∎

Given $a>1$ and $q \geq 1$ we will consider the operator

$$(3.7)\qquad \begin{aligned} L_{q,a}f(x) &= \int_0^\infty L_{q,a}(x,y)f(y)dy,\\ L_{q,a}(x,y) &= \chi_{(\frac{x}{a},ax)}(y)\frac{1}{\sqrt{xy}}\Big(1+\Big(\log\Big(1+\frac{xy}{|x-y|^2}\Big)\Big)^{1/q}\Big). \end{aligned}$$

It verifies the following lemma.

LEMMA 3.6: *Let $q \geq 1$ and $a>1$. The operator $L_{q,a}$ is bounded on $L^p(0,\infty)$ for $1 \leq p \leq \infty$.*

Proof: By the Marcinkiewicz interpolation theorem, it is enough to check that $L_{q,a}$ is bounded on $L^1(0,\infty)$ and in $L^\infty(0,\infty)$. Clearly, we have

$$\|L_{q,a}f\|_{L^1} \leq \int_0^\infty \|L_{q,a}(\cdot,y)\|_{L^1(dx)}|f(y)|dy \leq \sup_{y\in(0,\infty)} \|L_{q,a}(\cdot,y)\|_{L^1(dx)}\|f\|_{L^1}$$

$$\begin{aligned}\|L_{q,a}f\|_{L^\infty} &\leq \sup_{x\in(0,\infty)}\int_0^\infty |L_{q,a}(x,\cdot)|\|f(y)\|_{L^\infty}dy\\ &\leq \sup_{x\in(0,\infty)}\|L_{q,a}(x,\cdot)\|_{L^1(dy)}\|f\|_{L^\infty}.\end{aligned}$$

$L_{q,a}(x,y)$ is a symmetric function, and for $\alpha \in (0,1)$ such that $2\alpha/q < 1$ we get

$$\sup_{y\in(0,\infty)} \|L_{q,a}(\cdot,y)\|_{L^1} = \int_{1/a}^{a} \frac{1}{\sqrt{z}}dz + \int_{1/a}^{a} \frac{1}{\sqrt{z}}\Big(\log\Big(1+\frac{z}{|1-z|^2}\Big)\Big)^{1/q}\Big)dz$$
$$\leq C + C\int_{1/a}^{a}\Big(1+\frac{z}{|1-z|^2}\Big)^{\alpha/q}dz < \infty. \qquad \blacksquare$$

4. Proof of Theorem 2.1

By Theorem 3.4 in [1], R_μ is a principal value operator in $(0,\infty)$ in the sense of Definition 3.2, with associated kernel $R_\mu(x,y)$ given by

$$R_\mu(x,y) = x^{\mu+1/2}\int_0^\infty \frac{d}{dx}\Big(x^{-\mu-1/2}p_\mu(t,x,y)\Big)dt$$
$$= \frac{2\mu+1}{\pi}(xy)^{\mu+1/2}\int_0^\pi \frac{(y\cos z - x)(\sin z)^{2\mu}}{(x^2+y^2-2xy\cos z)^{\mu+3/2}}dz.$$

This kernel verifies (see Proposition 4.1 in [1]) $|R_\mu(x,y)| \leq C/|x-y|$. Hence, by using Theorem 3.5, in order to prove Theorem 2.1, it is enough to show

$$\Big|R_\mu(x,y) - \frac{1}{\pi}\frac{1}{x-y}\Big|\chi_{\{x/2<y<2x\}}(y) \leq CL(x,y),$$

where L is a kernel such that the operator $Lf(x) = \int_0^\infty L(x,y)f(y)dy$ is bounded on $L^p(0,\infty)$ for every $p \in [1,\infty)$. This inequality is a direct consequence of Lemma 3.6 and the following estimate:

$$R_\mu(x,y) = \frac{1}{\pi}\frac{1}{x-y} + O\Big(\frac{1}{x}\Big(1+\log_+\frac{\sqrt{xy}}{|x-y|}\Big)\Big), \quad x/2 < y < 2x.$$

See section 3 in [1] for the details.

5. Proof of Theorem 2.4

Let q satisfy $2 \leq q < \infty$.

5.1 Proof of Theorem 2.4 for $g^q_{\mu,t}$. Analogously to the classical case $g^q_{\mu,t}f(x) = \|t\partial_t P_{\mu,t}(f)(x)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)}$.

Lemma 5.1: *The vector-valued operator $t\partial_t P_{\mu,t}$ is given by the kernel*

$$t\partial_t p_\mu(t,x,y).$$

Moreover, $\|t\partial_t p_\mu(t,x,y)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)} \leq C/|x-y|$.

Proof: It is known (see [6, (16.4)]) that the Poisson kernel associated with Δ_μ, given by (2.2), has the following expression for $x,y,t\in(0,\infty)$:

(5.1)

$$p_\mu(t,x,y) = \frac{(2\mu+1)t}{\pi}(xy)^{\mu+1/2}\int_0^\pi \frac{(\sin z)^{2\mu}}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}}dz.$$

Let $f\in L^\infty_c(0,\infty)$. Define

$$g_\mu(t,x,y,z) = \frac{\partial}{\partial t}\Big(\frac{(2\mu+1)t}{\pi}(xy)^{\mu+1/2}\frac{(\sin z)^{2\mu}}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}}\Big).$$

For fixed t and x in $(0,\infty)$,

$$|g_\mu(t,x,y,z)f(y)| \leq \frac{C}{t^{2\mu+3}}(xy)^{\mu+1/2}(\sin z)^{2\mu}|f(y)| \in L^1((0,\infty)\times(0,\pi), dy\times dz).$$

This guarantees

$$\partial_t P_{\mu,t}f(x) = \int_0^\infty \partial_t p_\mu(t,x,y)f(y)dy,$$

where

(5.2)

$$\begin{aligned}\partial_t p_\mu(t,x,y) = &\\ &\frac{(2\mu+1)}{\pi}(xy)^{\mu+1/2}\int_0^\pi \frac{(\sin z)^{2\mu}dz}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}}\\ &-\frac{(2\mu+1)(2\mu+3)}{\pi}t^2(xy)^{\mu+1/2}\int_0^\pi \frac{(\sin z)^{2\mu}dz}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+5/2}}.\end{aligned}$$

The following estimate will be useful now and in the sequel. By the change of variable

$$u = z\sqrt{\frac{xy}{(x-y)^2+t^2}},$$

if $2\beta-\alpha>1$, we get

(5.3)

$$(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{z^{2\mu+\alpha}}{((x-y)^2+t^2+xyz^2)^{\mu+\beta}} \leq \frac{C}{\sqrt{xy}}\frac{1}{((x-y)^2+t^2)^{\beta-(\alpha+1)/2}},\quad t,x,y\in(0,\infty).$$

Then, by (5.3),

$$\begin{aligned}|t\partial_t p_\mu(t,x,y)| &\leq Ct(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{z^{2\mu}dz}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\\ &\leq \frac{Ct}{|x-y|^2+t^2}.\end{aligned}$$

Hence, $\|t\partial_t p_\mu(t,x,y)\|_{L^q(dt/t)} \le C|x-y|^{-1}$. ∎

The proof of Theorem 2.4 will be concluded by an application of Theorem 3.5. The former lemma applies to $S(x,y) = t\partial_t p_\mu(t,x,y)$, acting from $\mathbb{B}$ into $L^q_{\mathbb{B}}((0,\infty), dt/t)$, with norm less than $\|t\partial_t p_\mu(t,x,y)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)}$. By the estimates in (3.6), it remains to compare the kernels of g^q_t and $g^q_{\mu,t}$ in the region $x/2 < y < 2x$. This is done in the following lemma.

Lemma 5.2: *There exists a constant $C > 0$ such that*

$$\|t\partial_t p_\mu(t,x,y) - t\partial_t P_t(x-y)\|_{L^q((0,\infty),dt/t)}\chi_{(x/2,2x)}(y) \le CL_q(x,y),$$

where L_q is the kernel of a bounded operator on $L^p(0,\infty)$, $1 \le p < \infty$.

Proof: Clearly, by Lemma 5.1 and the estimates in (3.6), it is enough to see that there exists $a > 1$ with

$$\|t\partial_t p_\mu(t,x,y) - t\partial_t P_t(x-y)\|_{L^q((0,\infty),dt/t)}\chi_{(x/a,ax)}(y) \le CL_{q,a}(x,y), \tag{5.4}$$

where $L_{q,a}$ is defined in (3.7). The value of a is fixed in Step 3 below. We split the Poisson kernel as follows: $p_\mu = p_{\mu,1} + p_{\mu,2}$ by splitting the integral at $\pi/2$. Then we have

$$\begin{aligned} &t\partial_t p_{\mu,1}(t,x,y)\\ =&\frac{(2\mu+1)}{\pi}t(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}dz}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}}\\ &-\frac{(2\mu+1)(2\mu+3)}{\pi}t^3(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{(\sin z)^{2\mu}dz}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+5/2}}.\end{aligned}$$

Let us also consider the following kernels K_i, $i=1,2$. In the kernel $t\partial_t p_{\mu,1}$ we replace $\sin z$ by z and denote the corresponding kernel by $K_1(t,x,y)$, i.e.,

$$K_1(t,x,y) = \frac{2\mu+1}{\pi}t\partial_t\left(t(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{z^{2\mu}dz}{(|x-y|^2+t^2+2xy(1-\cos z))^{\mu+3/2}}\right).$$

The kernel

$$K_2(t,x,y) = \frac{2\mu+1}{\pi}t\partial_t\left(t(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{z^{2\mu}dz}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}\right)$$

is obtained from K_1 by replacing $1-\cos z$ by $z^2/2$.

For $0 < x/a < y < ax$, we split the left-hand side in (5.4) as

$$\begin{aligned}\|t\partial_t p_\mu(t,x,y) - t\partial_t P_t(x-y)\|_{L^q(dt/t)} &\le \|t\partial_t p_{\mu,1}(t,x,y) - K_1(t,x,y)\|_{L^q(dt/t)}\\ &+\|K_1(t,x,y)-K_2(t,x,y)\|_{L^q(dt/t)} + \|K_2(t,x,y) - t\partial_t P_t(x-y)\|_{L^q(dt/t)}\\ &+\|t\partial_t p_{\mu,2}(t,x,y)\|_{L^q(dt/t)} = A+B+C+D.\end{aligned}$$

Step 1: A: Since $\sin z \sim z$ and $|\sin z - z| \sim z^3$, for $z \in (0, \pi/2)$, by using the mean value theorem and then (5.3), for $0 < \varepsilon < 2\mu + 3$,

(5.5)
$$|t\partial_t p_{\mu,1}(t,x,y) - K_1(t,x,y)| \leq Ct(xy)^{\mu+1/2} \int_0^{\pi/2} \frac{z^{2\mu+2-\varepsilon} dz}{(|x-y|^2 + t^2 + xyz^2)^{\mu+3/2}} \leq \frac{Ct}{(xy)^{1-\varepsilon/2}(|x-y|^2+t^2)^{\varepsilon/2}}.$$

This inequality holds for every $t, x, y \in (0, \infty)$ with an absolute constant independent of t, x and y. In particular, if we choose $\varepsilon = (q-1)/q$ for $0 < t < \sqrt{xy}$ and $\varepsilon = (q+1)/q$ for $\sqrt{xy} < t < \infty$, we obtain

(5.6)
$$\begin{aligned} \|t\partial_t p_{\mu,1}(t,x,y) - K_1(t,x,y)\|^q_{L^q(dt/t)} &\leq C\Bigg[\int_0^{\sqrt{xy}} \frac{t^{q-1}}{(xy)^{(q+1)/2}(|x-y|^2+t^2)^{(q-1)/2}} dt \\ &+ \int_{\sqrt{xy}}^{\infty} \frac{t^{q-1}}{(xy)^{(q-1)/2}(|x-y|^2+t^2)^{(q+1)/2}} dt \Bigg] \\ &\leq C(xy)^{-q/2}. \end{aligned}$$

Step 2: B: Again, using $1 - \cos z \sim z^2/2$ and $|1 - \cos z - z^2/2| \leq Cz^4$, $z \in (0, \pi/2)$, for $\beta > 0$, by the mean value theorem

$$\left| \frac{1}{(|x-y|^2 + t^2 + 2xy(1-\cos z))^{\mu+\beta}} - \frac{1}{(|x-y|^2+t^2+xyz^2)^{\mu+\beta}} \right| \leq \frac{Cxyz^4}{(|x-y|^2+t^2+xyz^2))^{\mu+\beta+1}}.$$

From here it is easy to see that, for $0 < \varepsilon < 2\mu + 3$,

$$\begin{aligned} |K_1(t,x,y) - K_2(t,x,y)| &\leq C(xy)^{\mu+3/2} t \int_0^{\pi/2} \frac{z^{2\mu+4} dz}{(|x-y|^2+t^2+xyz^2)^{\mu+5/2}} \\ &\leq C(xy)^{\mu+1/2} t \int_0^{\pi/2} \frac{z^{2\mu+2-\varepsilon} dz}{(|x-y|^2+t^2+xyz^2)^{\mu+3/2}}. \end{aligned}$$

This is the estimate in (5.5), hence as in (5.6), we get

$$\|K_1(t,x,y) - K_2(t,x,y)\|_{L^q(dt/t)} \leq C/\sqrt{xy}.$$

Step 3: C: In this very part the comparison with the classical g-function takes place. For fixed x, t, and $y \neq x$, we have, by the change of variable

$$u = z\sqrt{\frac{xy}{(x-y)^2+t^2}},$$

$$K_2(t,x,y) = \frac{2\mu+1}{\pi} t\partial_t\Big(t(xy)^{\mu+1/2}\int_0^{\pi/2}\frac{z^{2\mu}}{((x-y)^2+t^2+xyz^2)^{\mu+3/2}}dz\Big)$$
$$= \frac{2\mu+1}{\pi} t\partial_t\Big(\frac{t}{|x-y|^2+t^2}\int_0^{\frac{\pi}{2}\sqrt{\frac{xy}{|x-y|^2+t^2}}}\frac{u^{2\mu}du}{(1+u^2)^{\mu+3/2}}\Big).$$

As

$$1 = (2\mu+1)\int_0^\infty \frac{u^{2\mu}}{(1+u^2)^{\mu+3/2}}du,$$

it implies

$$|K_2(t,x,y) - t\partial_t P_t(x-y)| = C\Big|t\partial_t\Big(\frac{t}{|x-y|^2+t^2}\int_{\frac{\pi}{2}\sqrt{\frac{xy}{|x-y|^2+t^2}}}^\infty \frac{u^{2\mu}du}{(1+u^2)^{\mu+3/2}}\Big)\Big|$$
$$\le \frac{Ct}{|x-y|^2+t^2}\int_{\frac{\pi}{2}\sqrt{\frac{xy}{|x-y|^2+t^2}}}^\infty \frac{u^{2\mu}du}{(1+u^2)^{\mu+3/2}}$$
$$+ \frac{C(xy)^{\mu+1/2}}{(|x-y|^2+t^2+xy)^{\mu+3/2}}$$
$$= N_1(t,x,y) + N_2(t,x,y).$$

By the same trick as the computation in (5.6),

$$\|N_2(.,x,y)\|_{L^q(dt/t)} \le C\Big(\int_0^\infty\Big|\frac{t}{|x-y|^2+t^2+xy}\Big|^q\frac{dt}{t}\Big)^{1/q} \le \frac{C}{\sqrt{xy}}.$$

To get the desired estimate for the kernel N_1, we write, for some positive $\delta_0 > 0$,

$$\|N_1(\cdot,x,y)\|_{L^q(dt/t)} \le \Big(\int_0^{\delta_0\sqrt{xy}}|N_1(t,x,y)|^q\frac{dt}{t}\Big)^{\frac{1}{q}} + \Big(\int_{\delta_0\sqrt{xy}}^\infty |N_1(t,x,y)|^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= I + II.$$

For the first part we would like to have $u^{2\mu}/(1+u^2)^{\mu+1/2}$ decreasing for

$$u > \frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}}$$

and $t \in (0, \delta_0\sqrt{xy})$. For negative μ this holds for every $t > 0$; for positive μ, $u^{2\mu}/(1+u^2)^{\mu+1/2}$ is decreasing if $u > \sqrt{2\mu}$. Hence it is sufficient that

$$u_0 = \frac{\pi}{2}\sqrt{\frac{xy}{(x-y)^2+t^2}} \ge \sqrt{2\mu},$$

i.e.

$$t \le \sqrt{(\frac{\pi}{2})^2\frac{1}{2\mu}xy - |x-y|^2}$$

whenever $t \leq \delta_0\sqrt{xy}$. The following lemma states the existence of such a δ_0 (this only makes sense for positive μ).

LEMMA 5.3: *For μ positive, there exist $a > 1$ and $\delta_0 = \delta_0(\mu, a)$ such that for $x/a \leq y \leq ax$, if $t \in (0, \delta_0\sqrt{xy})$, then*

$$t \leq \sqrt{\left(\frac{\pi}{2}\right)^2 \frac{1}{2\mu} xy - |x-y|^2},$$

Proof: We would like to have

$$t \leq \delta_0\sqrt{xy} \leq \sqrt{\left(\frac{\pi}{2}\right)^2 \frac{1}{2\mu} xy - |x-y|^2},$$

for $x/a \leq y \leq ax$. The inequality holds if and only if

$$x^2 + y^2 \leq \left(2 + \left(\frac{\pi}{2}\right)^2 \frac{1}{2\mu} - \delta_0^2\right) xy.$$

As $x/a \leq y \leq ax$, this is achieved if we can write

$$x^2 + y^2 \leq (1+a^2)x^2 \overset{?}{\leq} \frac{1}{a}\left(2 + \left(\frac{\pi}{2}\right)^2 \frac{1}{2\mu} - \delta_0^2\right) x^2 \leq \left(2 + \left(\frac{\pi}{2}\right)^2 \frac{1}{2\mu} - \delta_0^2\right) xy.$$

Thus, it is enough to choose $a > 1$ and $\delta_0 > 0$ with

$$a + a^3 \leq 2 + \left(\frac{\pi}{2}\right)^2 \frac{1}{2\mu} - \delta_0^2. \qquad \blacksquare$$

We fix a and δ_0 as in this lemma. Then in N_1,

$$\frac{u^{2\mu}}{(1+u^2)^{\mu+1/2}} \leq \frac{u_0^{2\mu}}{(1+u_0^2)^{\mu+1/2}} \leq \frac{\sqrt{|x-y|^2+t^2}}{(|x-y|^2+t^2+cxy)^{1/2}} \leq C\sqrt{\frac{|x-y|^2+t^2}{xy}},$$

for $u \geq u_0$.

Finally,

$$\begin{aligned} I^q &\leq C\int_0^{\delta_0\sqrt{xy}} \frac{t^{q-1}}{(xy)^{q/2}(|x-y|^2+t^2)^{q/2}} dt \leq C\int_0^{\delta_0\sqrt{xy}} \frac{t}{(xy)^{q/2}(|x-y|^2+t^2)} dt \\ &\leq C(xy)^{-q/2}\log\left(1+\frac{xy}{|x-y|^2}\right) \leq C(L_{q,a}(x,y))^q. \end{aligned}$$

For the second part,

$$\begin{aligned} II^q &\leq C\int_{\delta_0\sqrt{xy}}^\infty \frac{t^{q-1}}{(|x-y|^2+t^2)^q}\left(\int_0^\infty \frac{u^{2\mu}du}{(1+u^2)^{\mu+3/2}}\right)^q dt \leq C\int_{\delta_0\sqrt{xy}}^\infty \frac{dt}{t^{q+1}} \\ &\leq \frac{C}{(xy)^{q/2}}. \end{aligned}$$

STEP 4: D: Since for $z \in (\pi/2, \pi)$, $1 - \cos z \geq 1$, we have

$$\begin{aligned} |t\partial_t p_{\mu,2}(t,x,y)| &\leq Ct(xy)^{\mu+1/2} \int_{\pi/2}^{\pi} \frac{(\sin z)^{2\mu} dz}{(|x-y|^2 + t^2 + xy)^{\mu+3/2}} \\ &\leq \frac{Ct}{|x-y|^2 + t^2 + xy}, \end{aligned}$$

and therefore, by proceeding as in (5.6),

$$\begin{aligned} \|t\partial_t p_{\mu,2}(t,x,y)\|_{L^q(dt/t)} &\leq C\left(\int_0^\infty \frac{t^{q-1} dt}{(|x-y|^2 + t^2 + xy)^q} \right)^{1/q} \leq \frac{C}{\sqrt{xy}} \\ &\leq CL_{q,a}(x,y). \end{aligned}$$

5.2 PROOF OF THEOREM 2.4 FOR $g^q_{\mu,x}$. The proof of Theorem 2.4 for $g^q_{\mu,x}$ is a verbatim repetition of the proof for $g^q_{\mu,t}$ in the former section. We leave the details to the reader and we only sketch some steps. In analogy to the classical case, $g^q_{\mu,x} f(x) = \|tA_{\mu,x} P_{\mu,t} f(x)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)}$.

LEMMA 5.4: *The vector-valued operator $tA_{\mu,x}P_{\mu,t}$ is given by the kernel $tA_{\mu,x}p_\mu(t,x,y)$. Moreover, $\|tA_{\mu,x}p_\mu(t,x,y)\|_{L^q_{\mathbb{B}}((0,\infty),dt/t)} \leq C/|x-y|$.*

LEMMA 5.5: *There exists a constant $C > 0$ such that*

$$\|tA_{\mu,x}p_\mu(t,x,y) - t\partial_x P_t(x-y)\|_{L^q((0,\infty),dt/t)} \chi_{(x/2,2x)}(y) \leq CH_q(x,y), \tag{5.7}$$

where H_q is the kernel of a bounded operator on $L^p(0,\infty)$, $1 \leq p < \infty$.

Proof: It goes along the same line as the proof of Lemma 5.2, by splitting p_μ as $p_{\mu,1} + p_{\mu,2}$. Then we consider the kernels obtained by replacing successively $\sin z$ by z and $1 - \cos z$ by $z^2/2$ in the kernel $A_{\mu,x}p_{\mu,1}$. ∎

6. Proof of Theorem 2.5

Let us start by going from (i) to (ii).

PROPOSITION 6.1: *Let $f, g \in L^2((0,\infty), dx)$. Then*

$$\frac{1}{4} \int_0^\infty f(x)g(x)dx = \int_0^\infty \int_0^\infty t\frac{\partial}{\partial t} P_{\mu,t} f(x) t\frac{\partial}{\partial t} P_{\mu,t} g(x) \frac{dt}{t} dx. \tag{6.1}$$

Proof: Choose $f, g \in \mathcal{C}_c^\infty$. According to [9, Lemma 2, p. 23] we have

$$P_{\mu,t} f(x) = H_\mu(H_\mu(P_{\mu,t} f(\cdot)))(x) = H_\mu(e^{-tz} H_\mu f(z))(x).$$

Here H_μ denotes the Hankel transformation defined by [11, p. 127]

$$H_\mu(f)(x) = \int_0^\infty \sqrt{xy} J_\mu(xy) f(y)dy, x \in (0,\infty).$$

Since by [11, Th. 5.4-1] $x^{-\mu-1/2}H_\mu f$ is the Schwartz space and the function $\sqrt{z}J_\mu(z)$ is bounded in $(0,\infty)$, we can differentiate under the integral sign and we obtain

$$\frac{\partial}{\partial t} P_{\mu,t} f(x) = H_\mu(-ze^{-tz} H_\mu f(z))(x).$$

By Theorem 2.4, in the case $\mathbb{B} = \mathbb{R}$, $g^2_{\mu,t}$ is bounded on $L^2(0,\infty)$. Hence, we have

(6.2)

$$\int_0^\infty \int_0^\infty \left| t\frac{\partial}{\partial t} P_{\mu,t} f(x)\right| \left| t\frac{\partial}{\partial t} P_{\mu,t} g(x)\right| \frac{dt}{t} dx \leq \|g^2_{\mu,t}(f)\|_{L^2(0,\infty)} \|g^2_{\mu,t}(g)\|_{L^2(0,\infty)} < \infty.$$

Thus we can change the order of integration and, by using the Parseval equality for the Hankel transforms ([11, Th. 5.1-2]) we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty t\frac{\partial}{\partial t} P_{\mu,t} f(x) t\frac{\partial}{\partial t} P_{\mu,t} g(x) \frac{dt}{t} dx \\
&\quad = \int_0^\infty t \int_0^\infty H_\mu(-ze^{-tz}H_\mu f(z))(x) H_\mu(-ze^{-tz}H_\mu g(z))(x) dx dt \\
&\quad = \int_0^\infty t \int_0^\infty x^2 e^{-2xt} H_\mu f(x) H_\mu g(x) dx dt \\
&\quad = \int_0^\infty x^2 H_\mu f(x) H_\mu g(x) \int_0^\infty te^{-2xt} dt dx \\
&\quad = \frac{1}{4}\int_0^\infty H_\mu f(x) H_\mu g(x) dx = \frac{1}{4}\int_0^\infty f(x)g(x)dx.
\end{aligned}$$

The equality holds for any $f, g \in L^2(0,\infty)$ by density. ∎

Assume that $\mathbb{B}$ has Lusin type q, where $1 < q \leq 2$. Thus, its dual space $\mathbb{B}^*$ has Lusin cotype q', and by Theorem 2.4 we have $\|g^{q'}_{\mu,t} g\|_{L^{p'}(0,\infty)} \leq C\|g\|_{L^{p'}_{\mathbb{B}^*}(0,\infty)}$. On the other hand, (6.1) holds for any functions $f = \chi_A$, $g = \chi_B$ with A and B measurable sets of finite measure in $(0,\infty)$. Then (6.1) holds for simple functions $f \in L^p_{\mathbb{B}}((0,\infty),dx)$ and $g \in L^{p'}_{\mathbb{B}^*}((0,\infty),dx)$. Thus, by duality and Hölder we obtain $\|f\|_{L^p_{\mathbb{B}}(0,\infty)} \leq C\|g^q_{\mu,t} f\|_{L^p(0,\infty)}$ for simple functions, and by the density of these functions, we get it for any function $f \in L^p_{\mathbb{B}}((0,\infty),dx)$. This is the end of (i) implies (ii) of Theorem 2.5.

Now we shall prove (ii) implies (i) of Theorem 2.5 by using the next Proposition 6.2. Let $\mathbb{B}$ be a Banach space and $1 < q \le 2$. We define

$$\mathbb{A} = L^q_{\mathbb{B}}((0,\infty), dt/t).$$

A function in $\mathbb{A}$ is a two-variable function, namely $h(t,x)$, $t,x \in (0,\infty)$. Let us consider the operator

$$Q_\mu h(x) = \int_0^\infty t \int_0^\infty (\partial_t p_\mu(t,x,y))h(t,y)dy\frac{dt}{t}, \quad x \in (0,\infty),$$

for suitable functions h.

PROPOSITION 6.2: *Let $\mathbb{B}$ be a Banach space and $1 < p, q < \infty$. The operator $g^q_{\mu,t} \circ Q_\mu$ extends boundedly as an operator from $L^p_{\mathbb{A}}(0,\infty)$ into $L^p(0,\infty)$.*

In order to clarify the reading of the paper, we postpone the proof of this proposition until the end of the argument.

Proof of Theorem 2.5, (ii) implies (i): Let $1 < p < \infty$ and $1 < q \le 2$ such that

$$\|f\|_{L^p_{\mathbb{B}}(0,\infty)} \le C\|g^q_{\mu,t}(f)\|_{L^p(0,\infty)}.$$

Let $f \in L^{p'}_{\mathbb{B}}(0,\infty)$. We can choose $h \in L^p_{L^q_{\mathbb{B}}((0,\infty),dt/t)}$, with

$$\|h\|_{L^p_{L^q_{\mathbb{B}}((0,\infty),dt/t)}(0,\infty)} \le 1$$

and

$$\|g^{q'}_{\mu,t}(f)\|_{L^{p'}(0,\infty)} = \int_0^\infty \int_0^\infty t\frac{\partial}{\partial t}(P_{\mu,t}(f)(x))h(t,x)\frac{dt}{t}dx.$$

By changing the order of integration and taking into account the identity $p_\mu(t,x,y) = P_\mu(t,y,x)$, it follows that for good enough f and h,

$$\begin{aligned}
\|g^{q'}_{\mu,t}(f)\|_{L^{p'}(0,\infty)} &= \int_0^\infty \int_0^\infty t\frac{\partial}{\partial t}(P_{\mu,t}(f)(x))h(t,x)dtdx \\
&= \int_0^\infty \int_0^\infty t\partial_t\Big(\int_0^\infty p_\mu(t,x,y)f(y)dy\Big)h(t,x)dxdt \\
&= \int_0^\infty f(y)\int_0^\infty \int_0^\infty t\partial_t(p_\mu(t,x,y))h(t,x)dxdtdy \\
&= \int_0^\infty f(y)Q_\mu h(y)dy \le \|f\|_{L^{p'}_{\mathbb{B}^*}(0,\infty)}\|Q_\mu h\|_{L^p_{\mathbb{B}}(0,\infty)} \\
&\le C\|f\|_{L^{p'}_{\mathbb{B}^*}(0,\infty)}\|g^q_{\mu,t}(Q_\mu h)\|_{L^p(0,\infty)} \\
&\le C\|f\|_{L^{p'}_{\mathbb{B}^*}(0,\infty)}\|h\|_{L^p_{L^q_{\mathbb{B}}((0,\infty),dt/t)}(0,\infty)} \le C\|f\|_{L^{p'}_{\mathbb{B}^*}(0,\infty)},
\end{aligned}$$

where in the penultimate inequality we have used Proposition 6.2. Therefore, $\mathbb{B}^*$ has Lusin cotype q' and by [10, Corollary 2.6] we conclude that $\mathbb{B}$ has Lusin type q. ∎

Proof of Proposition 6.2: This proof mimics those of the boundedness of the g^q_μ functions and the Riesz transform. Let us write

$$g^q_{\mu,t}(\varphi)(x) = \|s\partial_s(P_{\mu,s}(\varphi)(x))\|_{L^q(ds/s)} = \|s\partial_s \int_0^\infty p_\mu(s,x,y)\varphi(y)dy\|_{L^q(ds/s)}.$$

Lemma 6.3: *Let $K_{s,t}(x,y)$ be the kernel of the operator: $h \longrightarrow s\partial_s(P_{\mu,s}(Q_\mu h))$. Then $\|K_{s,t}(x,y)\|_{(L^q((0,\infty),ds/s),L^{q'}((0,\infty),dt/t))} \le C/|x-y|$.*

Proof: Let h be a $\mathbb{B}$-valued continuous function with compact support in $(0,\infty)\times(0,\infty)$. Note that

$$s\partial_s(P_{\mu,s}(Q_\mu h)(x)) = \int_0^\infty s\partial_s p_\mu(s,x,y)\int_0^\infty\int_0^\infty \partial_t(p_\mu(t,y,z))h(t,z)dzdtdy.$$

Hence

$$K_{s,t}(x,z) = \int_o^\infty s\partial_s p_\mu(s,x,y)t\partial_t p_\mu(t,y,z)dy.$$

By the formula for kernels of a semigroup

$$\int_0^\infty p_\mu(s,x,y)p_\mu(t,y,z)dy = p_\mu(s+t,x,z),$$

hence

$$\begin{aligned} &K_{s,t}(x,z)\\ &\quad = st\partial_u^2 p_\mu(u,x,z)_{|u=s+t}\\ &\quad = st\partial_u^2\left[\frac{2\mu+1}{\pi}u(xz)^{\mu+1/2}\int_0^\pi \frac{(\sin v)^{2\mu}dv}{(|x-z|^2+u^2+2xz(1-\cos v))^{\mu+3/2}}\right]\Bigg|_{u=s+t}\\ &\quad = st\frac{2\mu+1}{\pi}(xz)^{\mu+1/2}\Bigg[-6(\mu+3/2)\\ &\qquad \times\int_0^\pi \frac{(s+t)(\sin v)^{2\mu}dv}{(|x-z|^2+(s+t)^2+2xz(1-\cos v))^{\mu+5/2}}\\ &\qquad + 4(\mu+3/2)(\mu+5/2)\int_0^\pi \frac{(s+t)^3(\sin v)^{2\mu}dv}{(|x-z|^2+(s+t)^2+2xz(1-\cos v))^{\mu+7/2}}\Bigg]\\ &\quad = (K^1_{s,t}+K^2_{s,t})(x,z). \end{aligned}$$

First, let us note that $|K^2_{s,t}(x,y)| \le C|K^1_{s,t}(x,y)|$. By (5.3)

$$|K^1_{s,t}(x,y)| \le \frac{Cst(s+t)}{(|x-y|^2+(s+t)^2)^2} \le C\frac{st}{(|x-y|+s+t)^3}, \quad s,t,x,y\in(0,\infty).$$

Therefore,

$$\int_0^\infty \left(\int_0^\infty |K_{s,t}(x,y)|^{q'} \frac{dt}{t}\right)^{q/q'} \frac{ds}{s} \leq \int_0^\infty \left(\int_0^\infty \left|\frac{st}{(|x-y|+s+t)^3}\right|^{q'} \frac{dt}{t}\right)^{q/q'} \frac{ds}{s}$$
$$\leq \frac{C}{|x-y|^q}. \qquad \blacksquare$$

In order to use Theorem 3.5, we consider $K_{s,t}(x,y)$ as the kernel of an operator

$$L^q_{\mathbb{B}}((0,\infty)\frac{ds}{s}) \to L^q_{\mathbb{B}}((0,\infty)\frac{dt}{t});$$

its norm is less than $\|K_{s,t}(x,y)\|_{(L^q((0,\infty),ds/s),L^{q'}((0,\infty),dt/t))}$. Then we introduce the analogue of Q_μ in the classical euclidean case. Following [5], we define

$$Q(h)(x) = \int_0^\infty t\partial_t P_t * h(\cdot,t)(x)\frac{dt}{t}, \quad x \in \mathbb{R}.$$

$Q(h)$ is well defined for functions h in a dense family of $L^p_{\mathbb{A}}(\mathbb{R})$, for instance, compactly supported functions on $\mathbb{R} \times \mathbb{R}_+$. We write

$$g^q_t(Qh)(x) = \|s\partial_s(P_s * (Qh))(x)\|_{L^q(ds/s)}.$$

In [5, Theorem 3.2] it is proved that $\|g^q_t(Q(h))\|_{L^p(\mathbb{R})} \leq C_{p,q}\|h\|_{L^p_{\mathbb{A}}(\mathbb{R})}$. Consequently, the map $h \mapsto g^q_t(Q(h))$ extends to a bounded map from $L^p_{\mathbb{A}}(\mathbb{R})$ to $L^p(\mathbb{R})$. We write

$$g^q_t(Qh)(x) = \|s\partial_s P_s * (Qh)(x)\|_{L^q_{\mathbb{B}}((0,\infty),ds/s)}.$$

The vector-valued operator $s\partial_s P_s * (Qh)(x)$ is given by the kernel

$$H_{s,t}(x,y) = \frac{1}{\pi}\frac{2st(s+t)((s+t)^2 - 3|x-y|^2)}{(|x-y|^2 + (s+t)^2)^3}.$$

Moreover, as $K_{s,t}(x,y)$, the kernel $H_{s,t}(x,y)$ defines a bounded operator from $L^q_{\mathbb{B}}((0,\infty),dt/t)$ into $L^q_{\mathbb{B}}((0,\infty),ds/s)$ with norm less than $C/|x-y|$.

Lemma 6.4: *There exists a constant $C > 0$ such that*

$$\|K_{s,t}(x,y) - H_{s,t}(x,y)\|_{L^q_{L^{q'}((0,\infty),dt/t)}((0,\infty),ds/s)}\chi_{(x/2,2x)}(y) \leq CM_q(x,y),$$

where M_q is the kernel of a bounded operator on $L^p(0,\infty)$, $1 \leq p < \infty$.

Proof: We can proceed as in the proofs of Lemmas 5.2 and 5.5: we write $K_{s,t} = K^1_{s,t} + K^2_{s,t}$ by splitting the integral in $K_{s,t}$ at $\pi/2$. We replace $\sin z$ by z in $K^1_{s,t}$ and then $1 - \cos z$ by $z^2/2$. $\blacksquare$

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